O(N) LINEAR SIGMA MODEL BEYOND THE HARTREE APPROXIMATION AT FINITE TEMPERATURE *

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We study the $O(N)$ linear sigma model with spontaneous symmetry breaking at finite temperature in the framework of the two-particle point-irreducible (2PPI) effective action. We go beyond the Hartree approximation by including the two-loop contribution, i.e., the sunset diagram. A phase transition of second order is found, whereas it is of first order in the one-loop Hartree approximation. Furthermore, we show the temperature-dependence of the variational mass parameters and comment on their relation to the physical sigma and pion masses.

1 Introduction

Nowadays it is common belief that the $O(N)$ linear sigma model has a second-order phase transition. Nevertheless, there is still recent progress on this subject as people develop new approximation methods of the effective potential at finite temperature.
Inspired by earlier investigations of the non-equilibrium properties of the model in the Hartree approximation[1] – where we found striking similarities to the situation in thermal equilibrium –, we wanted to know if a higher approximation is able to model the correct order of the phase transition.
Among recent publications there are, e.g., those by Bordag and Skalozub[2] and by Phat *et al.*[3] who used the 2PI CJT formalism with a "manually localized" Dyson–Schwinger equation and found a first-order phase transition even for higher loop-order. In contrast to that there is the 2PPI formalism developed by Verschelde *et al.*[4] which is able to mimic the correct order of the phase transition for plain Φ^4 theory[5].
We extend their 2-loop 2PPI analysis of plain Φ^4 theory to a full $O(N)$ model. For more details we refer the reader to a more comprehensive publication[6] and references therein.

*talk given by S. Michalski.

2 2PPI Effective Potential

The classical action of the $O(N)$ linear sigma model in 3+1 dimensions is given by

$$\mathcal{S}[\Phi] = \int d^4x \, \mathscr{L}[\Phi] = \int d^4x \left[\frac{1}{2}\partial_\mu \vec{\Phi} \cdot \partial^\mu \vec{\Phi} - \frac{\lambda}{4}\left(\vec{\Phi}\cdot\vec{\Phi} - v^2\right)^2 \right] , \quad (1)$$

where $\vec{\Phi} = (\Phi_1, \Phi_2, \ldots, \Phi_N)$ is an $O(N)$ vector. Using the formalism of the 2PPI effective action[1,4,5] we eventually find an expression for the temperature-dependent effective action (temperature times Helmholtz's free energy)

$$\Gamma_{\text{eff}}^{\text{2PPI}}[\phi_i, \mathcal{M}_{ij}^2] = \mathcal{S}[\phi] + \Gamma_{\text{class}}^{\text{2PPI}}[\mathcal{M}_{ij}^2] + \Gamma_q^{\text{2PPI}}[\phi_i, \mathcal{M}_{ij}^2] \ . \quad (2)$$

An intrinsic property of this formalism is that the propagator, which is used to evaluate the loop diagrams of the quantum corrections, is *naturally* a local one

$$G_{ij}^{-1}(p^2) = p^2 \ \delta_{ij} + \mathcal{M}_{ij}^2 \ .$$

The variation of the effective action with respect to the mass matrix $\mathcal{M}_{ij}^2$ yields the gap equation

$$\mathcal{M}_{ij}^2 = \lambda(\phi^2 - v^2) \ \delta_{ij} + 2\lambda \ \phi_i\phi_j + \hbar\lambda \ \left(\sum_\ell \Delta_{\ell\ell} \ \delta_{ij} + 2\Delta_{ij} \right) , \quad (3)$$

where $\hbar\Delta_{ij}$ is the local self-energy or the connected part of the expectation value of the local composite operator Φ^2. In Eq. (3) it enters as an explicit function of ϕ_i and $\mathcal{M}_{ij}^2$ given by the derivative

$$\Delta_{ij}(\phi_i, \mathcal{M}_{ij}^2) = 2 \ \frac{\delta\Gamma_q^{\text{2PPI}}[\phi_i, \mathcal{M}_{ij}^2]}{\delta\mathcal{M}_{ij}^2} \ . \quad (4)$$

Using an $O(N)$-invariant decomposition of the (variational) mass matrix $\mathcal{M}_{ij}^2$

$$\mathcal{M}_{ij}^2 = \frac{\phi_i\phi_j}{\vec{\phi}^2}\mathcal{M}_\sigma^2 + \left(\delta_{ij} - \frac{\phi_i\phi_j}{\vec{\phi}^2} \right)\mathcal{M}_\pi^2$$

and restricting $\vec{\phi} = (\phi, 0, \ldots, 0)$ we find the following gap equations

$$\mathcal{M}_\sigma^2 = \lambda\left[3\phi^2 - v^2 + 3\hbar\Delta_\sigma + (N-1) \ \hbar\Delta_\pi\right] \quad (5a)$$

$$\mathcal{M}_\pi^2 = \lambda\left[\phi^2 - v^2 + \hbar\Delta_\sigma + (N+1) \ \hbar\Delta_\pi\right] \ . \quad (5b)$$

The 2-loop approximation is used here so that the the quantum corrections of the effective potential consist of the 1-loop "log det" term plus the sunset graph. The graphs contributing to the effective action and to the local self-energy are displayed in Fig. 1.

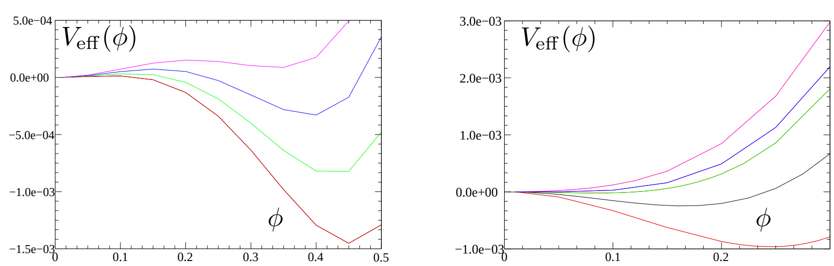
Figure 1. Graphical representation of the effective action and the self-energy in the two-loop approximation.

3 2-Loop Effective Potential at Finite Temperature

By inserting the (ϕ-dependent) solutions of the gap equations (5) into Eq. (2) the 1PI effective action is regained. The effective potential is then obtained by splitting off an irrelevant global factor of volume and temperature

$$V_{\rm eff}^{\rm 1PI}[\phi] = V_{\rm eff}^{\rm 2PPI}[\phi, \mathcal{M}_\sigma^2(\phi), \mathcal{M}_\pi^2(\phi)] \ .$$

We set $N = 4$, fix the renormalization scale to $\mu^2 = v^2$ and calculate the value of $V_{\rm eff}^{\rm 1PI}$ for fixed ϕ and temperature. In Fig. 2 we plot the effective potential for different temperatures both in the Hartree and in the 2-loop approximation. The potential in the Hartree approximation has a false vacuum – an evidence for a first-order phase transition – whereas the 2-loop effective potential shows only one minimum, i.e., a phase transition of second order. We call the (temperature-dependent) minimum of the 2-loop potential $\phi_0(T)$ and plot its value against temperature (cf. Fig. 3). There is a continous transition from a non-trivial vacuum for temperatures below the critical one to a phase with $\phi_0(T) = 0$ for temperatures $T > T_{\rm crit}$.



(a) Hartree approximation for temperatures $T/v = 1.46, 1.47, 1.48, 1.49$.

(b) 2-loop approximation for temperatures $T/v = 1.62, 1.66, 1.69, 1.70, 1.72$.

Figure 2. Effective 2PPI potential for $N = 4$, $\mu^2 = v^2$ and $\lambda = 1$. The potential clearly shows a first-order phase transition in the Hartree approximation whereas it is of second order in the 2-loop approximation.

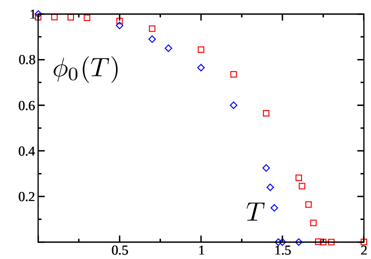


Figure 3. Temperature-dependence of the minimum $\phi_0(T)$ of the 2-loop effective potential. $\lambda = 1$ (squares), $\lambda = 0.1$ (diamonds). The continuous transition to zero is signal for a second-order phase transition.

At first sight this seems to be a contradiction to the result obtained by Bordag and Skalozub[2] who found a (weak) first-order phase transition beyond Hartree. However, they use the 2PI CJT approach to calculate the free energy which differs from ours. They obtain a local gap equation from the non-local Dyson–Schwinger equation by considering it in the zero momentum limit, which results in a loss of thermodynamic consistency because a local propagator does not follow from the 2PI formalism by a variational principle. It seems that, for some reason, this procedure changes the order of the phase transition which cannot be undone, not even by including infinitely many diagrams of higher order (see also,e.g., Ref. [3]).

4 Variational Mass Parameters and Physical Masses

4.1 Variational mass parameters

In Fig. 4 we show the values of the mass parameters $\mathcal{M}_\sigma$ and $\mathcal{M}_\pi$ as functions of temperature. Both of them show a smooth temperature-dependent behavior. They become equal for temperatures beyond the critical one where the symmetry is restored. The pion mass parameter is small but never equal to zero – even not for zero temperature. Though recalling Goldstone's theorem this should actually be the case. The violation of this theorem is due to the fact that – for practical reasons – one has to truncate the effective action at some level and therefore (slightly) violates symmetries of the underlying theory. For the full effective action at the stationary point there should be an equality between $\delta^2\Gamma/\delta\phi^2$ and $\mathcal{M}^2$ which is not present here, in the 2-loop approximation, as we will show in the following.

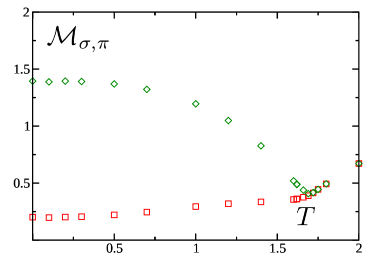


Figure 4. Temperature-dependence of the mass parameters for $\lambda = 1$. Diamonds: $\mathcal{M}_\sigma$, squares: $\mathcal{M}_\pi$.

4.2 Physical pion mass

We will define the *physical* masses as the eigenvalues of the matrix of second derivatives of the 1PI effective potential. This physical pion mass is *always* zero if the potential is $O(N)$-symmetric. This is obvious when the $O(N)$-symmetric physical mass matrix (with the eigenvalues $M^2_{\sigma,\mathrm{phys}}$ and $M^2_{\pi,\mathrm{phys}}$)

$$M^2_{ij,\mathrm{phys}} = \frac{\phi_i\phi_j}{\phi^2} M^2_{\sigma,\mathrm{phys}} + \left(\delta_{ij} - \frac{\phi_i\phi_j}{\phi^2}\right) M^2_{\pi,\mathrm{phys}}$$

is compared to the second derivatives of an $O(N)$-symmetric potential $V(\vec{\phi}^2)$

$$M^2_{ij,\mathrm{phys}} = \left.\frac{\partial^2 V(\vec{\phi}^2)}{\partial\phi_i\partial\phi_j}\right|_{\vec{\phi}=\vec{\phi}_0} = 2\delta_{ij}\, V'(\vec{\phi}^2) + 4\phi_i\phi_j\, V''(\vec{\phi}^2)\Big|_{\vec{\phi}=\vec{\phi}_0} .$$

Since $\vec{\phi}_0 = \{\phi_0(T), 0, \ldots, 0\}$ and $M^2_{\pi,\mathrm{phys}}$ is obtained by derivatives perpendicular to $\vec{\phi}_0$ it follows that

$$M^2_{\pi,\mathrm{phys}} = 2\, V'(\vec{\phi}_0^2) = 0 .$$

That is how Goldstone's theorem is saved in this context.

4.3 Physical sigma mass

The physical sigma mass is then given by the second derivative of the 1PI effective potential

$$M^2_{\sigma,\mathrm{phys}}(T) = \left.\frac{\partial^2 V^{\mathrm{1PI}}_{\mathrm{eff}}(\phi)}{\partial\phi^2}\right|_{\phi=\phi_0(T)} .$$

We calculated this number numerically fitting the (numerically obtained) effective potential with a polynomial. The results are shown in Fig. 5.

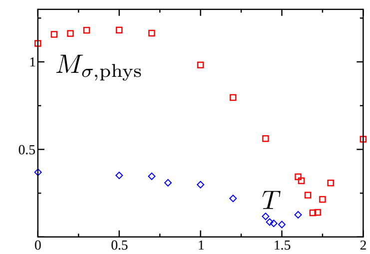


Figure 5. Physical σ mass as a function of temperature. $\lambda = 1$ (squares), $\lambda = 0.1$ (diamonds).

5 Summary and Outlook

We have analyzed the $O(N)$ linear sigma model in the 2PPI formalism within the 2-loop approximation, i.e., beyond leading order (or Hartree). As in the $N = 1$ version of the model[5] the phase transition, which is of first order in the Hartree approximation, becomes a second-order one. Furthermore there is no direct contradiction to the results of Bordag and Skalozub[2] and Phat *et al.*[3] since they use a different method.
As in the Hartree approximation and in the $N = 1$ version we find that the variational mass parameter of the pion quantum fluctuations is different from zero in the broken symmetry phase, so that a naive particle interpretation of this quantity – suggested by the large-N analysis – becomes problematic. By defining the physical sigma and pion masses as the eigenvalues of the matrix of second derivatives of the 1PI effective potential, Goldstone's theorem is saved.
A future step could certainly be to extend this analysis to non-equilibrium, i.e., to investigate if the system exhibits a second-order phase transition in the 2-loop approximation out of equilibrium.

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